Reachability Analysis of Closed Loop Switching Power Converters

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***Abstract*—A design verification method for closed-loop switching power converters is presented in this paper. The method computes the set of reachable states from an initial set of states with non-deterministic parameters. This is demonstrated with buck converters in closed-loop configurations. We model the buck-converter as a switched linear system and the controller as a linear system. The circuit with a simple hysteresis controller is modeled as well. The method is automatic and uses the hybrid systems reachability analysis tool SpaceEx. The applications and limitations of the tool are explored in this study.**

I. Introduction

When designing a switched-mode converter, a designer may rely on computerized analysis such as simulation. A variety of software tools exist for such modeling and simulation, including Simulink/Stateflow, LabView, Plexim PLECS, PSpice, among others. Such analysis is indispensable during the design process, as it aids the designer by giving them a first-pass view of whether the converter operates as expected. The "'expected operation"' may be based on the designer's experience and intuition, or it may be according to a design specification, such as the input/output currents and voltages, operating temperature ranges, expected manufacturing variations in components, etc.

However, while simulations aid the designer in such first-pass analysis, they are inherently *incomplete*, in the sense that one simulation run corresponds to a single execution of the system. That is, such analysis can at best provide a counterexample that the system does not behave correctly, but cannot prove that every execution of the system operates according to the specification (due to an infinite number of possible initial conditions, component variations taking values in the reals, etc.). Additionally, while some of these tools have the capability to model the converter controller as software (e.g., Simulink/Stateflow or LabView), they generally do not do so, and tools like PSpice provide only circuit-level simulations and have no capability to analyze the way the controller will actually be implemented in a modern system—via software running on a digital computer.

This paper describes a general reachability-based method for verifying closed-loop systems, applied in particular to switched-mode power supplies. We model the converter and controller as switched linear systems, and compute an overapproximation of the set of reachable states of the system, which are any states that may be visited by following the dynamics of the system from any initial condition (of which there may be uncountably many). The difference between reachability analysis and simulation is that every run of reachability overapproximates all possible executions of the system, whereas simulation would model one. Thus, if reachability is sound, in the sense that if the reachable states (or overapproximations thereof) do not violate a property, then the system does not violate the property. We use the hybrid systems [8], [7] verification tool SpaceEx for computing the reachable states [3], although there are a variety of tools that could be used [2] and have similar modeling frameworks. The limitations here are that reachability computations are expensive compared to simulation, and that the analysis is model-based and thus subject to any imperfections of the model. A reachability method for switched-mode power converters, which relies on the ellipsoidal toolbox [6], was presented in [4]. Another reachability method using SpaceEx was applied to open-loop verification of buck-converters and multilevel converters in [5].

II. SpaceEx

SpaceEx is a verification platform for hybrid systems. So, given a mathematical model of a hybrid system, SpaceEx verifies, or ensures beyond reasonable doubt, that the system satisfies all the desired safety properties. Essentially, it is used to compute the sets of reachable states of the system. It is not just a single tool but a development platform on which many different verification algorithms are implemented. Currently, PHAVer, which applies to linear hybrid automata, and LGG Support Function scenario, which implements a variant of the Le Guernic Girard algorithm, are utilized. SpaceEx is composed of a model editor, analysis core, and a web interface, as illustrated in Figure1. The model editor is a graphical editor for creating the models out of nested components. The analysis core is a command line program that takes the model file (in .xml format) and a configuration file - which specifies initial states and scenarios. Subsequently, it analyzes the system and produces desired output files. Lastly, the web interface is

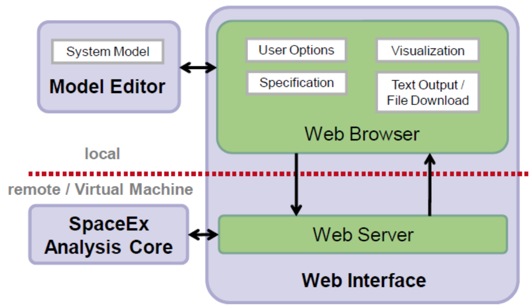

Fig. 1. SpaceEx software architecture.

a GUI in which the user can specify the initial states and various parameters, run the core, and visualize the output files graphically. It is browser based and accesses the core through a web server that can be running remotely or locally on a virtual machine. The reachability algorithm operates on symbolic states, which is the cross product of a set of discrete states (locations) and continuous states (variable valuations). Since the reachability for hybrid automata is undecidable and not guaranteed to terminate, a few options are available to control the algorithm. These include setting a number of maximum iterations and setting relative and absolute error.

III. Model

A. Closed-Loop Buck Converter

A buck converter is a switchmode step-down DC to DC converter that is comprised of two switches (typically a transistor and a diode) and an inductor and a capacitor, as shown in Figure 2. The switches alternate between connecting the inductor to source voltage to store energy in the inductor and discharging the inductor into the load. The frequency of switching and the duty cycle, which refers to the ratio of the period when the inductor is being charged, control the operation of the circuit, along with input voltage. In an open-loop configuration, the switching frequency and duty cycle are fixed, but are variable (depending on control strategy) in a closed-loop system. In this particular study, the closed-loop buck converter is of primary concern.

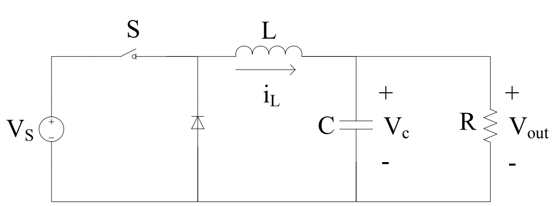

Fig. 2. Buck converter circuit.

The buck circuit, in continuous conduction, has two modes. One when the switch (transistor) is open and the inductor is discharging, and one when the switch is closed, with the inductor charging. To begin the derivation of the closed-loop buck converter system, it is useful to first study how the open-loop system is modeled. The circuit can be modeled as a switched affine (linear with fixed input) system of the form:

$$\dot{x}_{\sigma(t)} = A_{\sigma(t)}x + B_{\sigma(t)},$$

where for each $i \in M, A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^n$, and $\sigma(t) : \mathbb{R} \to M$ is a function mapping time to either closed switch or open switch mode. The capacitor voltage, V_c and the inductor current i_L are state variables of the system.

$$x = \begin{bmatrix} i_L \\ V_c \end{bmatrix} \tag{1}$$

For both modes, the circuit system matrix can be modeled as follows:

$$A_o = A_c = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}. \tag{2}$$

where the A_o matrix is when the the circuit when the switch is open and A_c matrix is when the circuit is when the switch is closed. However, the affine input term is different for the two modes. For the closed switch, the presence of the source voltage must be accounted for, and thus:

$$B_c = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_s \tag{3}$$

Conversely, for the open switch mode, the source voltage is not connected and results in the following vector:

$$B_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix} V_s \tag{4}$$

With feedback control, the converter output is measured and, subsequently, used to adjust operation (often duty cycle) to obtain desired result. If input to the circuit was 5V with reference voltage of 2V, the buck circuit will continue to adjust the duty cycle until the output voltage matches the desired reference voltage of 2V. Utilizing this error signal, more accurate results can be obtained than in an open-loop configuration. Therefore, a stabilizing controller in frequency domain was designed using pole placement. The controller design was adopted from Matlab/Simulink switched-mode power converter models by COPEC [1]. The equivalent linear system controller state space components are shown below:

$$A_{ctrl} = \begin{bmatrix} -\frac{1}{p_1} & 0 & 0 \\ -\frac{p_2}{p_1 p_3} + \frac{1}{p_3} & -\frac{1}{p_3} & 0 \\ -\frac{p_2 p_4}{p_1 p_3 p_5} + \frac{p4}{p3p5} & \frac{-p4}{p_3 p_5} + \frac{1}{p_5} & 0 \end{bmatrix} \tag{5}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B_{ctrl} = \begin{bmatrix} \frac{1}{p_1} \\ \frac{p_2}{p_1 p_3} \\ \frac{p_4 p_2}{p_1 p_3 p_5} \end{bmatrix}$$

where each p_i is a real constant chosen such that the controller is stabilizing. Now, the feedback system is described as two interconnected linear systems, one of the plant—i.e., the buck converter—and one of the controller. The plant has two states, and the controller has three states. These two systems are linked by an error term, e, which is the difference between the reference voltage, (V_{ref}), and output, (V_{out}), voltages. That is, $e = V_{ref} - V_{out}$ and $V_{out} = V_c$, therefore $\dot{e} = -\dot{V}_c$. This error term must be factored into the model, as converter adjusts its duty cycle according to the error value. The composed model must behave as follows:

$$\dot{x} = A_c \cdot x_c + B_{comp} \left(V_{ref} - V_{out}\right) \quad (6)$$

where B is either B_c, for the closed switch mode, or B_o, for the open switch mode. After algebraic simplification, the final composed switched affine system modeling the closed-loop buck controller with the plant, controller, and error term has five states and two modes. The system is:

$$A_c = \begin{bmatrix} 0 & \frac{-1}{L} & 0 & 0 & 0 & 0 \\ \frac{1}{C} & \frac{-1}{RC} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{p_1} & -\frac{1}{p_1} & 0 & 0 \\ 0 & 0 & -\frac{p_2}{p_1 p_3} & -\frac{p_2}{p_1 p_3} + \frac{1}{p_3} & -\frac{1}{p_3} & 0 \\ 0 & 0 & -\frac{p_4 p_2}{p_1 p_3 p_5} & -\frac{p_2 p_4}{p_1 p_3 p_5} + \frac{p_4}{p_3 p_5} & -\frac{p_4}{p_3 p_5} + \frac{1}{p_5} & 0 \end{bmatrix} \quad (7)$$

$$\dot{x_c} = \begin{bmatrix} \dot{i_L} \\ \dot{V_c} \\ \dot{x1} \\ \dot{x2} \\ \dot{x3} \end{bmatrix} \quad B_c = \begin{bmatrix} \frac{V_s}{L} \\ 0 \\ \frac{1}{p_1} \cdot V_{ref} \\ \frac{p_2}{p_1 p_3} \cdot V_{ref} \\ \frac{p_4 p_2}{p_1 p_3 p_5} \cdot V_{ref} \end{bmatrix} \quad B_o = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{p_1} \cdot V_{ref} \\ \frac{p_2}{p_1 p_3} \cdot V_{ref} \\ \frac{p_4 p_2}{p_1 p_3 p_5} \cdot V_{ref} \end{bmatrix}$$

The controller stabilizes the plant by switching between the open and closed modes based on the value of the controller state in relation to the reference voltage periodically (i.e., it determines the duty-cycle of pulse-width modulation). This

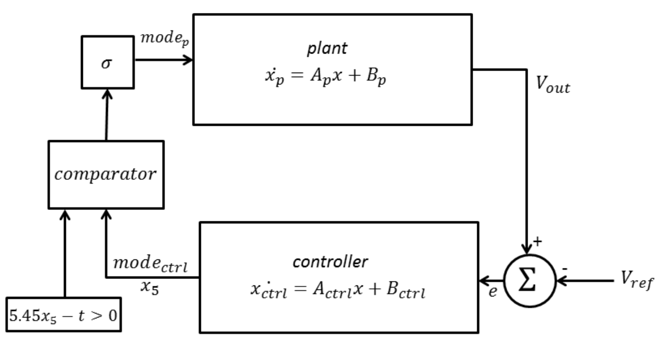

Fig. 3. Block diagram of buck converter and linear controller system.

system was tested in Simulink with an input voltage of $5V$ and a reference of $5V$, and was observed to operated as expected. The result is shown below: The capacitor voltage stabilizes around $2V$, as shown in the second plot in Figure 4. All other parameters of the composed system functioned as expected as well. The inductor current and controller states reach steady-state and the PWM signal illustrates obvious switching. Thus, the composition of the buck converter and linear controller system is sound.

IV. SpaceEx Analysis

A. Hysteresis Controller

To first test closed-loop modeling capability of SpaceEx, a hysteresis controller was implemented with the buck converter model. This type of controller is a self-oscillating feedback

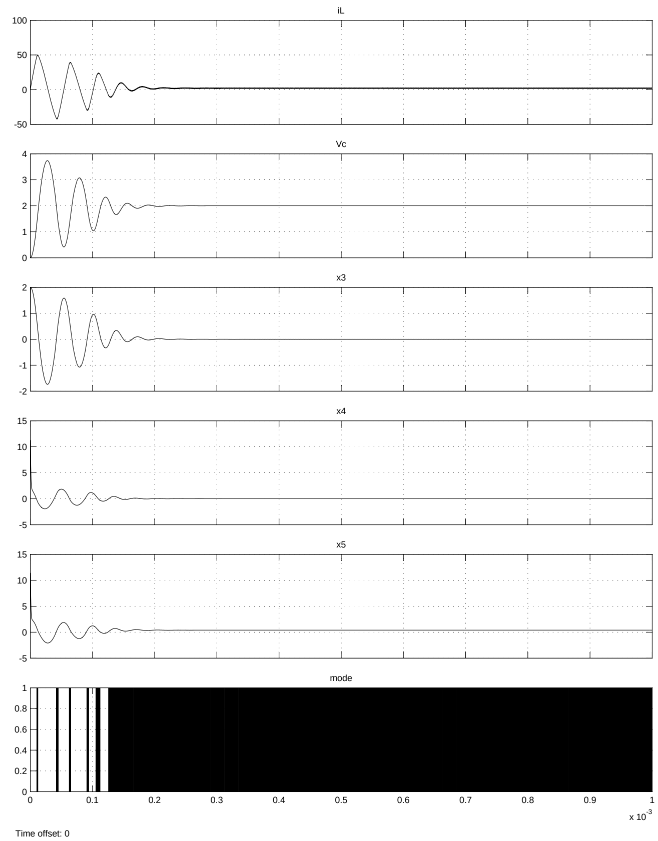

Fig. 4. Simulink output for the state variables and PWM signal.

controller that switches abruptly between two states. Essentially, a control is restricted to be between a lower and an upper bound. In this case, the two states are closed-switch (charging) and open-switch (discharging) and the capacitor voltage, V_c, is controlled between bounds $V_{ref} - \delta$ and $V_{ref} + \delta$, where δ is a predetermined constant. The hybrid system model is shown in Figure 5.

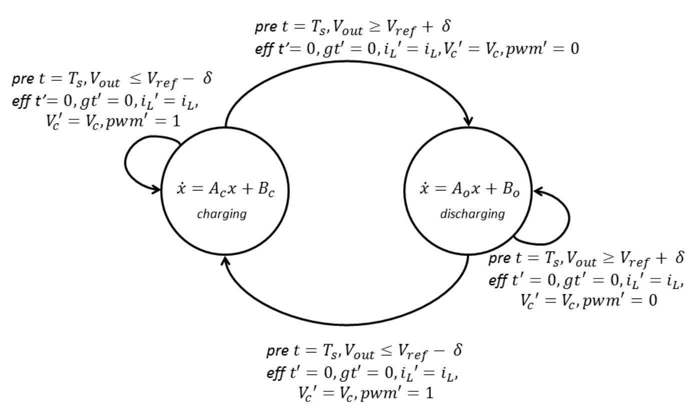

Fig. 5. Hybrid model of the buck converter with a hysteresis controller.

This simplified closed-loop buck converter system was modeled in SpaceEx with $\delta = 0.005$ and the following results were achieved: As shown in Figure 6 , after receiving

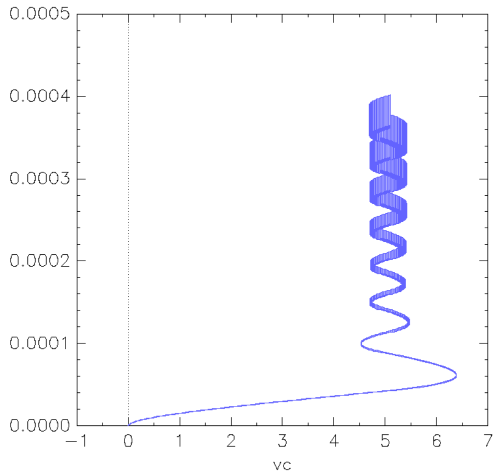

Fig. 6. Output capacitor voltage, V_c (V) vs. time, t(s), from SpaceEx.

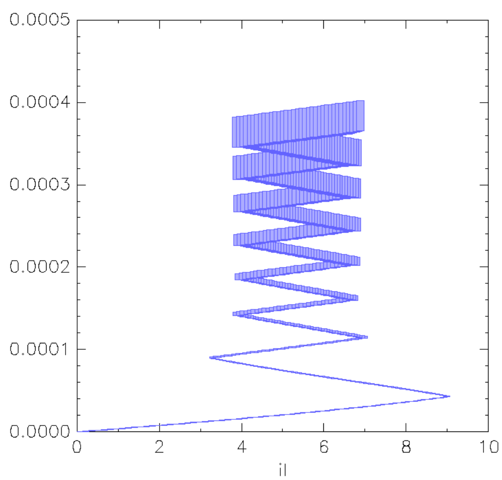

Fig. 7. Output inductor current, i_L (A) vs. time, t(s), from SpaceEx.

an input voltage of 12V, the capacitor voltage eventually settles down to a value around 5V. The inductor current also begins to stabilize, as seen in Figure 7. SpaceEx computes an overapproximation of the set of reachable states of the buck converter system, which are dependent on the dynamics of the system from specific initial states. Parameters V_c, t ,i_L and gt (global time) were initialized at 0 and V_s = 12V. The system is set to be in the charging (switch-closed) mode in its initial state. This reachability analysis was determined to be sound, as the capacitor voltage remains within reasonable bounds around 2V, which is the expected behavior of the circuit. The inductor current also stabilizes within reasonable bounds. Compared to a traditional simulation, all possible executions were overapproximated, not just one in particular. This allows one to conclude that if reachability is attained for a specific property, the system will, also, always achieve that property. For the switching buck converter, the capacitor voltage, V_c, being "bucked" down to a lower reference voltage of 5V from a source voltage of 12V was the property of interest. The overapproximation of the reach states, thus, allows a closed-loop buck circuit with a hysteresis controller to be effectively modeled in SpaceEx.

B. Linear Controller

Nonetheless, a hysteresis controller is not the standard method of controlling a buck converter, as it simplifies the system dramatically. However, it was effective in testing the closed-loop modeling capability of SpaceEx. Since the test proved to be successful, the linear controller was implemented with the buck converter system. When this model was run in SpaceEx, no switching occured, and the capacitor voltage and inductor current failed to stabilize. These results are shown below in Figures 8 and 9. Since the composed system was tested

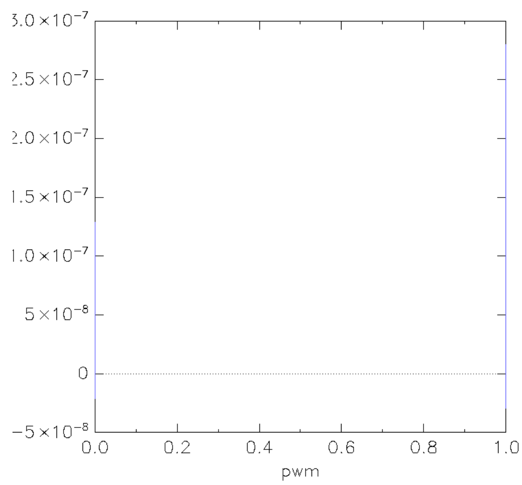

Fig. 8. PWM vs. Global Time.

in Simulink and found to be correct, the flawed results could stem from either how the system was modeled in SpaceEx or intrinsic SpaceEx limitations. The overapproximation of reach states may not be enabling the switching between the charging and discharging modes. The correct plot for the PWM signal would be similar to Figure 4. This, in turn, would hinder the buck converter operation of decreasing the capacitor voltage. The .xml model also may not be capturing the correcting switching behavior within the transitions. Stricter guards and invariants may be necessary to allow the switching to occur. Presently, the transition from states is enabled when one of the controller state variables, x_5, is determined essentially to be positive or negative. As shown in the block diagram in Figure 3, this result is utilized to choose the plant mode.

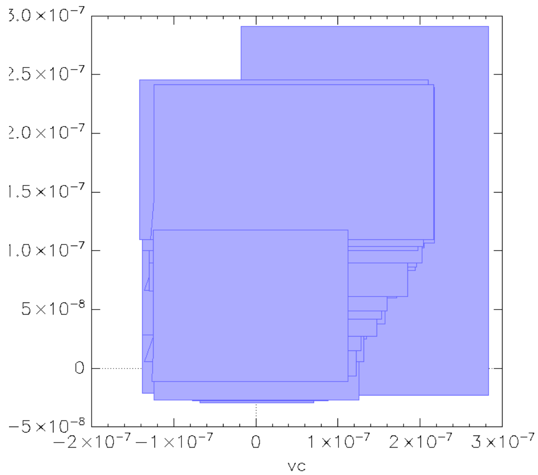

Fig. 9. Capacitor Voltage vs. Global Time.

V. Conclusion and Future Work

Applying hybrid systems reachability tool, SpaceEx, proved to be advantageous in the verification of the open-loop buck converter configuration as well as the closed-loop hysteresis controller model. The reachability analysis performed on the equivalent switched affine systems provided valuable information on the behavior of the converter circuits. Both systems exhibited proper operation of a buck converter, with the capacitor voltage decreased to a certain voltage (reference voltage in the case of the hysteresis system) and all other parameters also stabilizing and behaving as expected. SpaceEx computes an overapproximation of the set of reachable states of the system and, thus, ensures beyond reasonable doubt that the system satisfies all the desired safety properties for all possible executions. Therefore, both the open-loop system and hysteresis controller system satisfy all the desired properties and can be deemed as robust designs.

However, for the buck converter and linear controller system, the SpaceEx model did not run as expected. Switching between charging and discharging modes did not occur and the capacitor voltage and other parameters failed to reach steady-state. This could arise from the problems with how the system was modeled or overapproximation issues within SpaceEx itself. For future work, the model will be explored further to determine if any other constraints can be added to enable switching. If not, additional tests will be performed to investigate this limitation of SpaceEx.

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