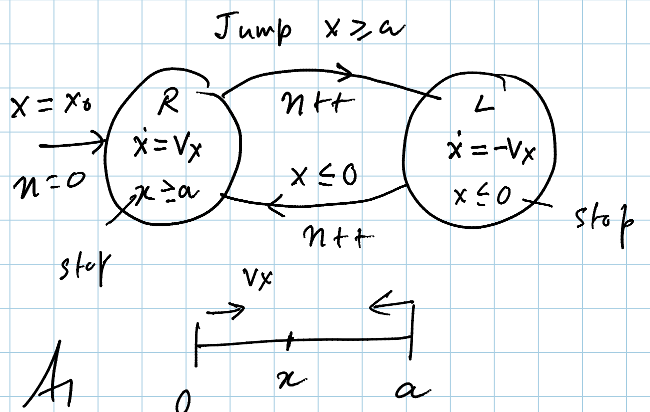
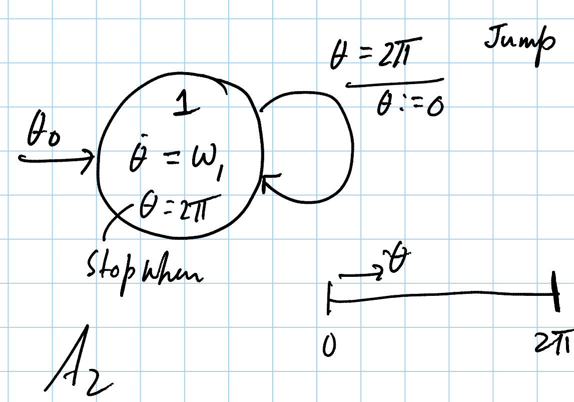
Tuesday, October 23, 2012
11:03 AM

HW1 Problem 2

1D billiard ball



1D - Satellite



A_1 & A_2 are bisimilar

A_1 implements A_2 and vice verse

There Exists a Forward Sim Rel from A_1 to A_2

$\text{type}(x) = [0, a]$

$\text{type}(loc) = \{R, L\}$

$Q_1 \triangleq val(\{x, loc\})$

$Q_2 \triangleq val(\{\theta, loc_2\})$

R_1 : Candidate FSR $\quad R_1 \subseteq Q_1 \times Q_2$

$$R_1 \triangleq \{(s, u) \mid s \in Q_1,\ u \in Q_2,\ s.x = \frac{a}{2\pi} u.\theta\}$$

Rough calc $\quad \boxed{\frac{x}{a} = \frac{\theta_1}{2\pi}} \quad x = \frac{a}{2\pi}\theta \quad \boxed{v_x = \frac{a}{2\pi}\omega_1}$

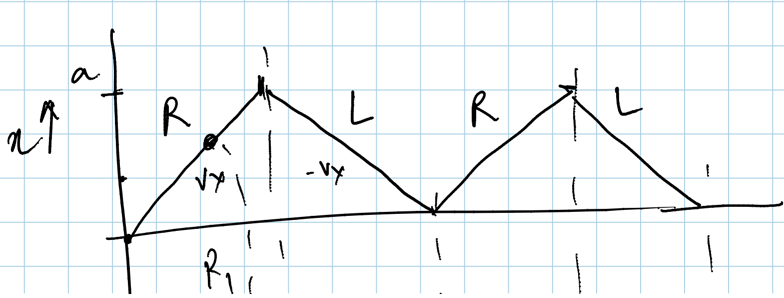
Assumtions

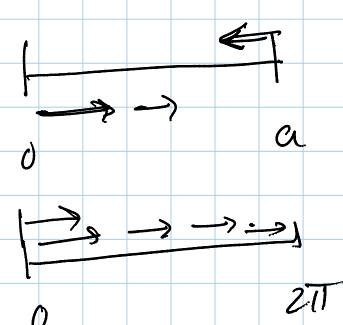
(1)

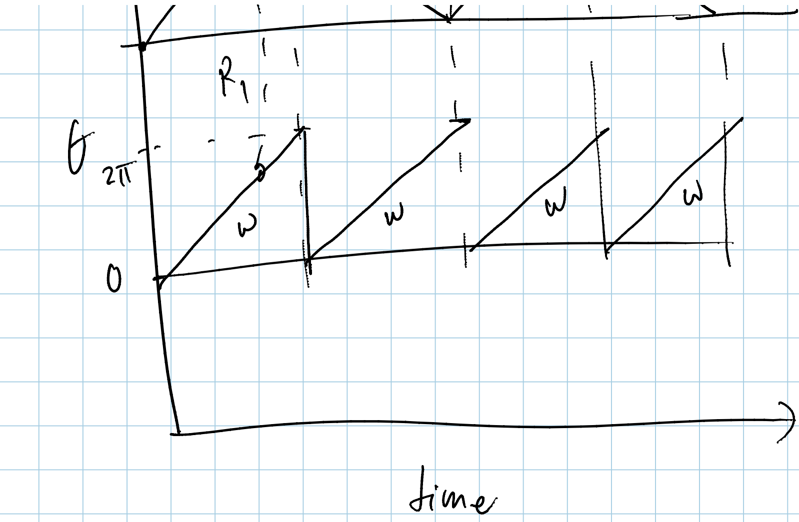
$x_0 = \frac{a}{2\pi}\theta_0 \quad (2)$

Claim: R_1 is a FSR from A_1 to A_2.

Proof:



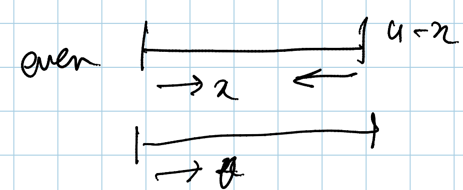




$Q_1 = val(\{n, x, loc\})$

$Q_2 = val(\{\theta, loc\})$

$$R_2 = \left\{ (s, u) \;\middle|\; \begin{array}{l} s.x = \text{even} \Rightarrow \frac{a}{2\pi}\theta = x \quad \wedge \\ s.x = \text{odd} \Rightarrow \frac{a}{2\pi}\theta = a - x \end{array} \right\} \quad \begin{array}{l} \text{— (3)} \\ \text{— (4)} \end{array}$$



Proof

Base

inductive Step

(a) $s \xrightarrow{\tau_1} s'$ and $\boxed{s \, R_2 \, u}$

we show $\boxed{\exists \tau_2, u'}$ s.t $u \xrightarrow{\tau_2} u' \;\wedge\; s' R u'$

τ_2 uniquely defined by u and $\tau_2.\text{ltime}$

$\tau_2.\text{ltime} \triangleq \tau.\text{ltime}$, say $= t$

$$\left. \begin{array}{l} s'.x = s.x + V_x \cdot t \\ u'.\theta = u.\theta + \omega_1 \cdot t \end{array} \right\} \text{ by def of } A_1 \, A_2$$

Case (i) $s.n = \text{even}$

$$u'.\theta = u.\theta + \omega_1 t$$
$$= \frac{2\pi}{a} s.x + \omega_1 t \quad \text{[From } R_2 \text{ (3)]}$$

$= \ldots 2\pi \ldots$ [From Assumptio

$= \frac{2\pi}{a} s.x + w_1 t$ [ ]

$= \frac{2\pi}{a} s.x + \frac{2\pi}{a} . V_x . t$ [From Assumption (1)]

$= \frac{2\pi}{a} \left[s.x + V_x . t \right]$

$= \frac{2\pi}{a} \left[s'.x \right]$

$s'.x =$ even so u' and s' satisfies R_2 (3)

Case (ii) $s.x =$ odd

Next we show that γ_2 is indeed allowed from u.

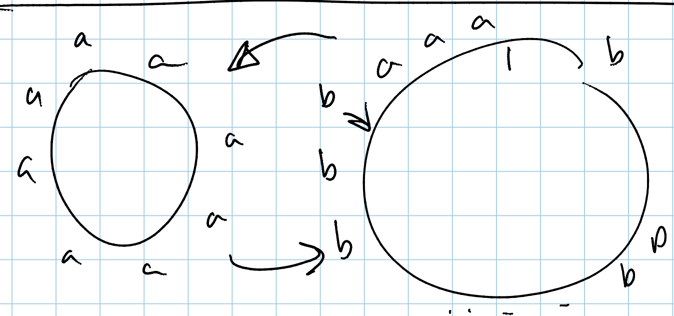
$\forall t' \in \gamma_2 . dom$ $\gamma_2(t')$ does not satisfy the stopping cond ~~$\theta \leq 2\pi$~~ $\theta < 2\pi$

$\gamma_2(t').\theta = u.\theta + t'.w_1$

$= \frac{2\pi}{a} (\underbrace{s.x + t'.V_x}_{< a})$ [using R_2]

$< \frac{2\pi}{a} . a = 2\pi$

(b) $s \xrightarrow{bounce} s'$ and $s R u$ bound we show $\exists u'$ s.t $u \to u'$ $s' R u'$



Stability

Hybrid automaton A_1

Recall, an Execution of A is alternating sequence

$\alpha = \tau_0 a_1 \tau_1 a_2 \cdots$ $\quad \tau_i : [0,t] \to val(X)$

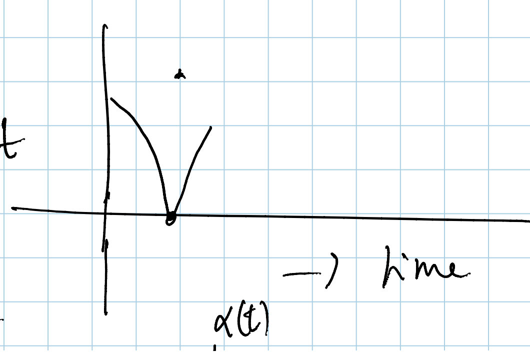
$a_i \in A$

$\tau_i.lstate \to \tau_{i+1}.fstate$

$\alpha(t) \quad t \in [0, \alpha.ltime]$ $\quad \alpha.ltime = \sum_i \tau_i.ltime$

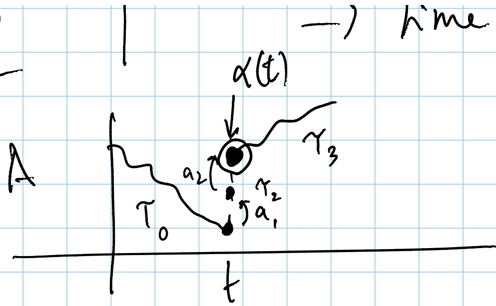
$\alpha(t) \triangleq \alpha'.lstate$

when α' is the longest prefix of α with $\alpha'.ltime \leq t$



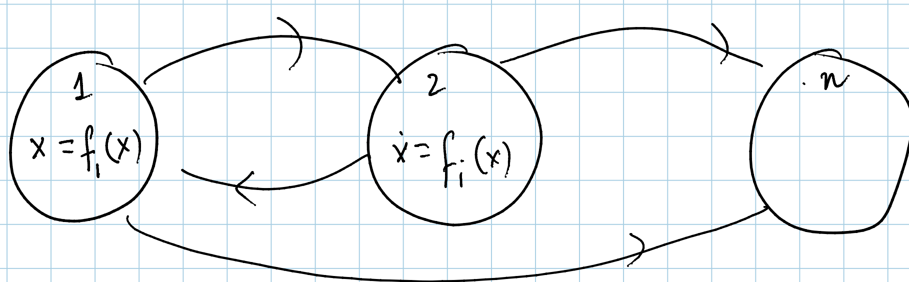
Stability is a property of

Stability is a property of the continuous variables of A



$\alpha(t)$: valuation of the continuous variables

$|\alpha(t)|$: norm of the valuation of the continuous variables at time t.



$I = \{1, 2, \ldots n\}$

$\forall i \in I \quad f_i : \mathbb{R}^n \to \mathbb{R}$

$\dot{x} = f_i(x) \to$ subsystems

$\dot{x} = k \quad \dot{x} = Ax$

Assumption : $\forall i \in I \quad f_i(0) = 0$

The origin is an equilibrium point for each subsystem

Def [Lyapunov Stability].

The origin is a stable equilibrium point of A in the sense of Lypunov if $\forall \ \varepsilon > 0 \ \exists \ \delta_1 = \delta_1(\varepsilon) > 0$ such that

[for every closed execution fragment α of A
if $|\alpha(0)| \leq \delta_1 \Rightarrow \forall t \leq \alpha.\text{ltime} \ |\alpha(t)| \leq \varepsilon$.

$x \in \mathbb{R}^2$

Remarks if $\Theta_A = \delta$-Ball

δ-Ball invariants

$\delta_1(\varepsilon)$

Remarks If $\partial U A$

then ε-Ball invariants for free