State Estimation of Dynamical Systems with Unknown Inputs: Entropy and Bit Rates

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Motivation: Harrier Jump Jet\(^{(1)}\)

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -g \sin \theta_1 - \frac{c}{m} x_2 + \frac{u_1}{m} \cos \theta_1 - \frac{u_2}{m} \sin \theta_1, \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= g (\cos \theta_1 - 1) - \frac{c}{m} y_2 + \frac{u_1}{m} \sin \theta_1 + \frac{u_2}{m} \cos \theta_1, \\
\dot{\theta}_1 &= \theta_2, \\
\dot{\theta}_2 &= \frac{r}{J} u_1.
\end{align*}
\]

State variables: \(x_1, x_2, y_1, y_2, \theta_1, \theta_2\)
Input variables: \(u_1, u_2\)

What is the bit rate between the jet and the command center needed to estimate the state of the jet up to an \(\epsilon\) error?

Problem Setup

- $f$ globally Lipschitz in both arguments with constants $L_x$ and $L_u$
- $K$ a compact set in $\mathbb{R}^n$
- $\mathcal{U}$ a set of input signals (will be defined next) in $\mathbb{R}^m$

What is the minimum number of bits per second needed to estimate the state of the system up to an $\varepsilon$ error.
Related Work

• Entropy and Minimal Data Rates for State Estimation and Model Detection
  [Liberzon and Mitra, HSCC’16, CDC’16, TAC’17]
  • Autonomous dynamical systems (no inputs), exponential convergence of error
• Optimal Data Rate for State Estimation of Switched Nonlinear Systems
  [Our paper in HSCC’17]
  • Switched systems, finite number of modes, combination between constant
    and exponentially converging error

[A. V. Savkin. Automatica’06, F. Colonius. SIAM’12, M. Rungger and M. Zamani. HSCC’17...]
Space of Input Signals: $\mathcal{U}(\eta, \mu, u_{max})$

Given $\eta, \mu$ and $u_{max} \geq 0$, if $u: [0, \infty) \to \mathbb{R}^m$ belongs to $\mathcal{U}(\eta, \mu, u_{max})$, then $\forall t, \tau \geq 0, ||u(t) - u(t + \tau)|| \leq \mu \tau + \eta$ and $|u(t)| \leq u_{max}$.
Entropy Definition

A function of the system dynamics and the allowed estimation error that lower bounds the needed bit rate of the channel
Approximating Functions

$z: \mathbb{R}^+ \to \mathbb{R}^n$ is an $(\varepsilon, T)$-approximating function for $\xi(x_0, u, t)$ if:

$\|\xi(x_0, u, t) - z(t)\| \leq \varepsilon$ for all $t \in [0, T]$. 
Approximating Sets and Entropy

\( \hat{\mathcal{Z}} = \{ \hat{z}_1, \hat{z}_2, \ldots, \hat{z}_M \} \) is a \((T, \varepsilon, K)\)-approximating set if: \( \forall x_0 \in K \) and \( u \in \mathcal{U}(\eta, \mu, u_{max}) \), there exists \( \hat{z}_i \in \hat{\mathcal{Z}} \) that is \((\varepsilon, T)\)-approximating for \( \xi(x_0, u, t) \) over \([0, T]\).

\( s_{est}(T, \varepsilon, K) \) is the minimum cardinality of such approximating set.

Entropy:

\[
h_{est}(\varepsilon, K) := \limsup_{T \to \infty} \frac{1}{T} \log s_{est}(T, \varepsilon, K).
\]
State Estimation Algorithms with Fixed Bit Rates

\[
x_0 \in K \quad u \in U \quad \dot{x} = f(x, u)
\]

Encoder/Sensor

\[\xi(x_0, u, iT_p) \to b \text{ bits} \to \] Finite-bandwidth Channel

\[b \text{ bits} \to \] Decoder/Estimator

\[z(t) \approx \xi(x_0, u, t)\]

Average bit rate: \(b_r(\varepsilon, K) = \frac{b}{T_p}\).
Theorem 1: There is no algorithm with fixed bit rate that, for any \( x_0 \in K, u \in \mathcal{U}(\eta, \mu, u_{max}) \) and \( T \geq 0 \), constructs an \((\varepsilon, T)\)-approximating function for the trajectory \( \xi(x_0, u, \cdot) \) while achieving a bit rate less than the entropy of the system.

Proof: Existence of such algorithm implies the existence of an \((\varepsilon, T, K)\)-approximating set of cardinality smaller than \( s_{est} \), contradiction.
Upper Bound on Entropy of Nonlinear Systems

Computing entropy is hard. Computing bounds is easier.
Distance between trajectories after $t$ seconds

\[ M_x = nL_x + \frac{1}{2} ; \quad M_u = m\sqrt{mL_u} \]

\[ \|x - x'\|^2 \leq e^{2M_xt} (\|x - x'\|^2 + M_u^2 \int_0^t \|u(s) - u'(s)\|^2 ds) \]
Lemma 1: If $\delta_u$, $\delta_x$, and $T_p$ are small enough, the output of the construction is an $(\varepsilon, T)$-approximating trajectory of the given trajectory.
\[(K, \varepsilon, T) \text{—approximating set construction}\]

- As \(x_0\) and \(u\) vary, what is the number of functions that can be constructed by the algorithm in the previous slide?
- At the first step, the construction chooses:
  - One from \(\left(\frac{\text{diam}(K)}{2\delta_x}\right)^n\) possible quantization points
  - One from \(\left(\frac{u_{\max}}{\delta_u}\right)^m\) possible quantization points
- At each time step of size \(T_p\), the construction chooses:
  - One from \(\left(\frac{\varepsilon}{\delta_x}\right)^n\) possible quantization points in the state space
  - One from \(\left(\frac{u_{\max}}{\delta_u}\right)^m\) possible quantization points in the input space
- There are \(\left\lfloor\frac{T}{T_p}\right\rfloor\) time steps
(K, \varepsilon, T) — approximating set construction

The number of functions that can be constructed by the procedure is upper bounded by:

\[
\left( \frac{\text{diam}(K)}{2\delta_x} \right)^n \left( \frac{u_{\max}}{\delta_u} \right)^m \left( \left( \frac{\varepsilon}{\delta_x} \right)^n \left( \frac{u_{\max}}{\delta_u} \right)^m \right)^{\frac{T}{T_p}}
\]
Intermediate upper bound on entropy

Substituting the bound on the cardinality of the approximating set in the entropy definition while fixing $\delta_x = \frac{\varepsilon}{2} e^{-M_x T_p}$, leads to:

$$h_{est}(\varepsilon, K) \leq \frac{2nM_x}{\ln 2} + \frac{1}{T_p} \left( n \log 2 + m \log \frac{u_{max}}{\delta_u} \right)$$

As $u_{max} \to 0$, $h_{est}(\varepsilon, K) \leq \frac{nM_x}{\ln 2}$, similar the earlier bound $\frac{nL_x}{\ln 2}$ by Liberzon and Mitra in HSCC’16 (remember $M_x \leq nL_x + \frac{1}{2}$).
Entropy upper bound

Fix an $\varepsilon > 0$, $h_{est}(\varepsilon, K)$ is upper bounded by:

$$\frac{2nM_x}{\ln 2} + \frac{1}{\min\{\rho(\mu, \eta, \varepsilon), \frac{1}{M_x}\}} \left( n \log 2 + m \log \frac{u_{max}}{\eta} \right),$$

where $\rho(\mu, \eta, \varepsilon) = \frac{2\eta}{\mu} \left( -1 + \left( 1 + \left( \frac{\varepsilon}{M_{ue}} \right)^2 \frac{9\mu}{32\eta^3} \right)^{\frac{1}{3}} \right)$ and $M_x = nL_x + \frac{1}{2}$

Remember:

$u_{max}$ is the maximum $|| \cdot ||_\infty$ of the input signal,

$||u(t) - u(t + \tau)||_\infty \leq \mu \tau + \eta$ and

$n$ and $m$ are the state and input dimensions.
Upper bound discussion

- It increases *quadratically* with $\eta$
- It increases as $\frac{1}{1 - O(\mu)}$ with $\mu$
- It increases *logarithmically* with $u_{max}$
- It increases as $\Omega(\varepsilon^{-\frac{2}{3}})$ as $\varepsilon$ goes to zero
Back to the Harrier Jump Jet Example

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
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\dot{\theta}_1 &= \theta_2 \\
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State variables: \( x_1, x_2, y_1, y_2, \theta_1, \theta_2 \)
Input variables: \( u_1, u_2 \)

What is the bit rate between the jet and the command center needed to estimate the state of the jet up to an \( \epsilon \) error? \( h_{\text{est}}(\epsilon, K) \leq 61 \text{ Kbps} \) (when \( \epsilon = 0.5, \mu = 10, \eta = 20 \))
Bound variation as $\mu$ and $\eta$ change

When only small variation of input with large jumps is allowed:

For $\mu = 0.1$ and $\eta = 45$, $h_{est}(\varepsilon, K) \leq 255$ Kbps

When large variation of input with only small jumps is allowed:

For $\mu = 20$ and $\eta = 0.1$, $h_{est}(\varepsilon, K) \leq 2.6$ Kbps
Upper Bound on Entropy of Systems with Linear Inputs
Systems with Linear Inputs

\[ \dot{x} = f(x) + u, \]

where \( x_0 \in K \), a compact set in \( \mathbb{R}^n \) and \( u \in \mathcal{U}(\eta, \mu, u_{max}) \).
Distance between trajectories after $t$ seconds

\[
\|x - x'\| \leq e^{l_{xt}} \left( \|x - x'\| + \int_0^t \|u(s) - u'(s)\|ds \right)
\]
Use the same \((\varepsilon, T)\)-approximating function construction

Lemma 1: If \(\delta_u, \delta_x\) and \(T_p\) are small enough, the output of the construction is an \((\varepsilon, T)\)-approximating trajectory of the given trajectory.
Entropy upper bound

Fix an $\varepsilon > 0$, $h_{est}(\varepsilon, K)$ is upper bounded by:

$$\frac{2nM_x}{\ln 2} + \frac{1}{\min\{\rho(\mu, \eta, \varepsilon), \frac{1}{L_x}\}} \left( n \log 2 + m \log \frac{u_{max}}{\eta} \right),$$

where $\rho(\mu, \eta, \varepsilon) = \frac{2\eta}{\mu} \left( -1 + \sqrt{\left( 1 + \frac{\mu \varepsilon}{4\eta} \right)} \right)$. 
Pendulum example

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-Mgl}{I}\sin x_1 + \frac{u}{I}
\end{align*}
\]

where \(Mgl = 0.98, I = 1, u_{max} = 2, \mu = 0.1, \eta = 1\) and \(\epsilon = 0.5\).

Using the upper bound on entropy:

- For general nonlinear systems: \(h_{est}(\epsilon, K) \leq 1386\ Kbps\),
- For systems with linear inputs: \(h_{est}(\epsilon, K) \leq 0.6\ Kbps\).

The latter bound can be much tighter than the former.
Ongoing Work

Relating the bounds in this paper with previous bounds on switched systems
Thank you