On Differential Privacy of Distributed Control System

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General Question

- For distributed control systems, how expensive is it to preserve privacy?

- Navigation
  - Routing delays vs location privacy
- Smart Grid
  - Peak demand vs schedule privacy

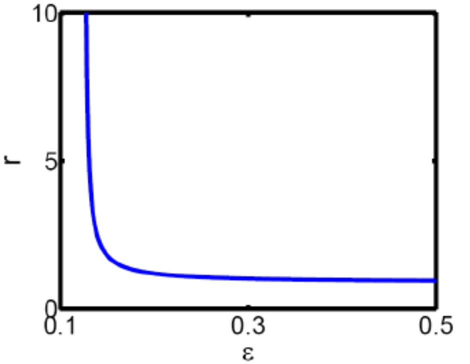
Differential Privacy (DP)

- Introduced in [1]: a private mechanism should not provide substantially different outputs if one users data changes

- In [2]: minimization of estimation error for open-loop dynamical systems with differential privacy

- [3] discuss cost of privacy for consensus algorithms

[1] C. Dwork et al. TCC2006
[2] JL. Ny and GJ Pappas. TAC2014
[3] Z. Huang et al. WPES2012

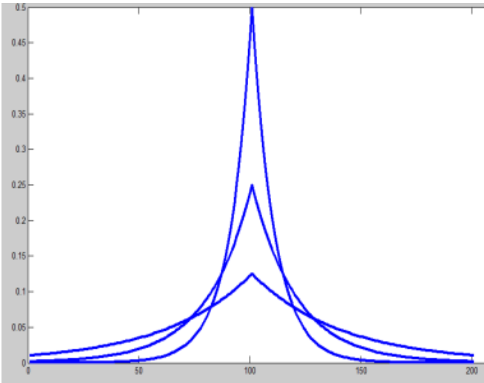


Differential Privacy (DP)

- **Def. DP**: M is a mechanism that gives **ε-differential privacy** with $\epsilon > 0$, if for all datasets D and D' that differ in one user's data, for all subset of observations $S \subseteq Range(M)$,
$$\Pr[M(D) \in S] \leq e^{\epsilon} \Pr[M(D') \in S]$$

A Simple Example of DP Algorithm

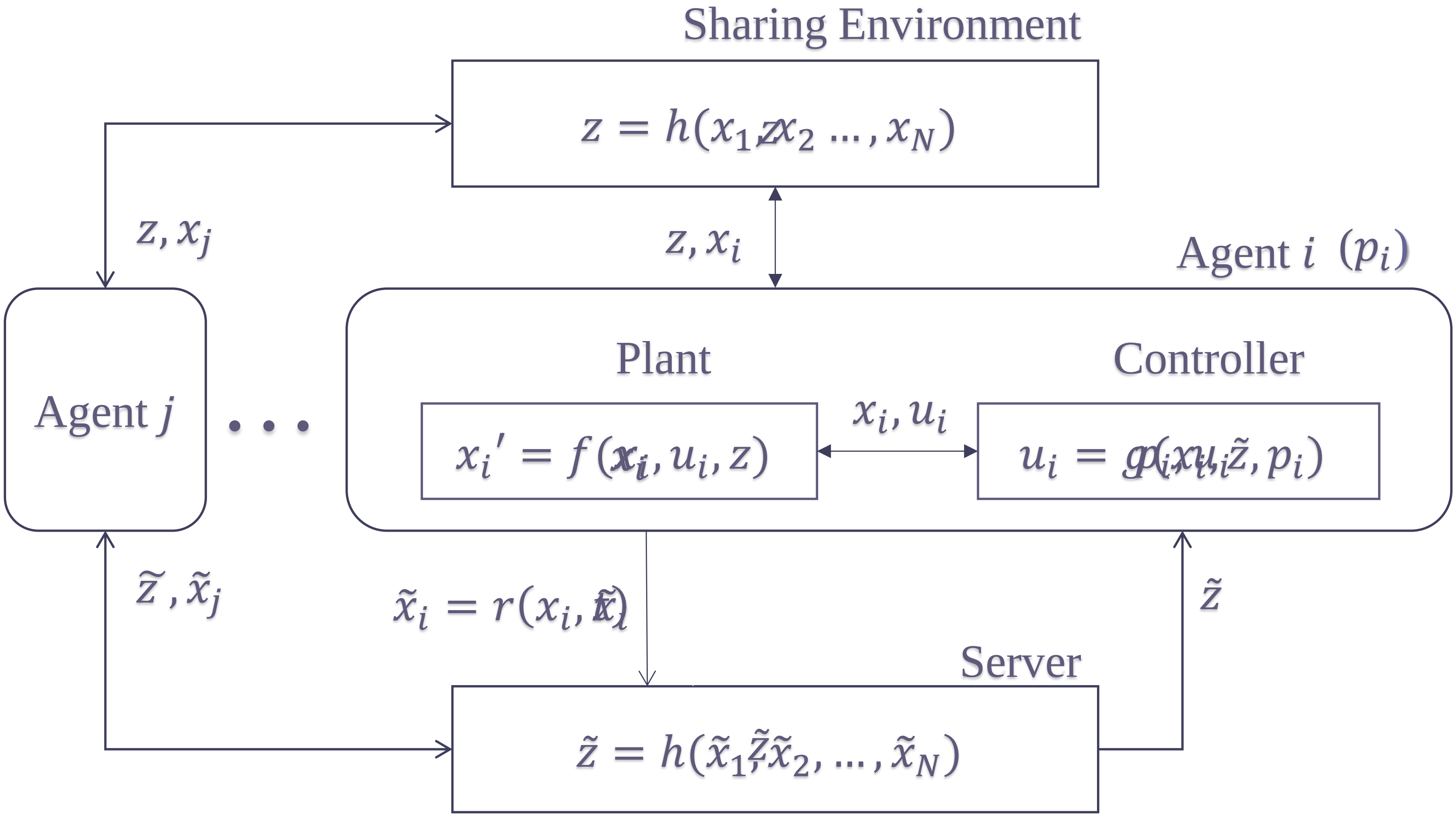
- Weight watchers example
  - Multiple people diet together. Each people participated has a weight value in the range $[a, b]$. They want to compute the average weight without reporting their exact weight.
  - Each participant will add a carefully chosen noise to his own weight and report it to the server.
  - The server can then publish the average without bleaching the individuals' privacy



  - Laplace noise
  - Noise $\sim Lap(\epsilon(b-a))$

$$f(x) = \frac{1}{2\epsilon(b-a)} e^{\epsilon(b-a)}$$

Distributed Control System



Observables: $\tilde{x}_i, \tilde{z}$. **Valuable information**: p_i

Example: Navigation

- Routing of N agents on a 2-D plan:
  - The agent i's state $x_i \in \Re^2$
  - Preference of agent i is a path with length T: $p_i \in \Re^{2T}$
  - The environment state, center of mass: $z = \frac{1}{N}\sum_i x_i$
  - The update law of the individual agent's state at time t:
    $$x_i(t+1) = x_i(t) + c(z(t) - x_i(t)) + u_i(t)$$
  - The control law:
    $$u_i(t) = -cz_i(t) + 0.8(p_i(t) - x_i(t))$$

- To design: the report strategy:
  $$\tilde{x}_i(t) = r(t, x_i(t))$$

DP for Distributed Control System

- **The sensitive data of the system $\boldsymbol{p = \{p_1, p_2, \dots, p_N\}}$**
  - **$\boldsymbol{p_i}$ is the desired trajectory of agent $\boldsymbol{i}$:**

$$\boldsymbol{p_i = [p_i(0), p_i(1), \dots, p_i(T)]}$$

Unbounded change in p_i results in unbounded change in system's behavior

- **Def. DP:** Let Obs be any observation stream of bounded time T and p, p' be different in agent i's preferences. The DCS is ϵ-DP if:

$$\Pr[\ p \text{ leads to } Obs\ ] \leq \cancel{e^{\varepsilon}} \Pr[\ p' \text{leads to } Obs\ ]$$

$$e^{\epsilon |p_i - p_i'|}$$

Cost of Privacy for Distributed Control System

- **Average Cost:** $\frac{1}{\mathrm{N}}\sum_{t=0}^{T}\sum_i |x_i(t) - p_i(t)|^2$
  - Fixed a DCS, depends only on the preferences p

- Baseline Mechanism M': $\boldsymbol{\widetilde{x}_i(t) = x_i(t)}$

- The **Cost of Privacy** of a DP mechanism M **is:**

$$\boldsymbol{CoP} = \sup_{p} \mathbf{E}[\boldsymbol{Cost_{M,p} - Cost_{M',p}}]$$

Sensitivity

- The sources of uncertainty of the system
  - The preferences of agents
  - The randomized report function
- Fixed a sequence of observation (Obs) and the agents' preferences (p), the trajectories of all the agents are fully specified: $x(Obs, p, t)$
- **Def. Sensitivity**: difference in the system's states resulting from change in individual's p_i

$$\Delta(t) = \sup_{Obs, adj(p,p')} \frac{|x(Obs,p,t) - x(Obs,p',t)|}{|p_i - {p_i}'|}$$

A DP Algorithm

- Theorem: The following distributed control system is ϵ-differentially private:
  - At each time t each agent adds an vector of independent Laplace noise $Lap(\frac{\Delta(t)T}{\epsilon})$ to its actual state:

$$\widetilde{x}_i(t) = x_i(t) + Lap(\frac{\Delta(t)T}{\epsilon})$$

- Sensitivity and Cost of Privacy are properties of the dynamics of the system.

Linear Distributed Control Systems

- Linear distributed control system:

$$z_i(t) = \frac{1}{N}\Sigma_i x_i$$
$$x_i(t+1) = Ax_i(t) + cz(t) + u_i(t)$$
$$u_i(t) = -c\tilde{z}(t) + K(x_i(t) - p_i(t))$$

- We will design an ϵ-differentially private mechanism for this system and reason about cost of privacy.

Sensitivity of Linear Distributed Control Systems

- **Sensitivity**:

  $$\Delta(t) = |(cI + K)^t| + |\sum_{s=0}^{t}(cI + K)^s(I - K)|$$

  - Independent to the number of agents.
  - Converges to a constant if the closed-loop dynamics is stable.
  - Diverges exponentially otherwise.

- DP Mechanism:

  $$\widetilde{x_i}(t) = x_i(t) + Lap(\frac{\Delta(t)T}{\epsilon})$$

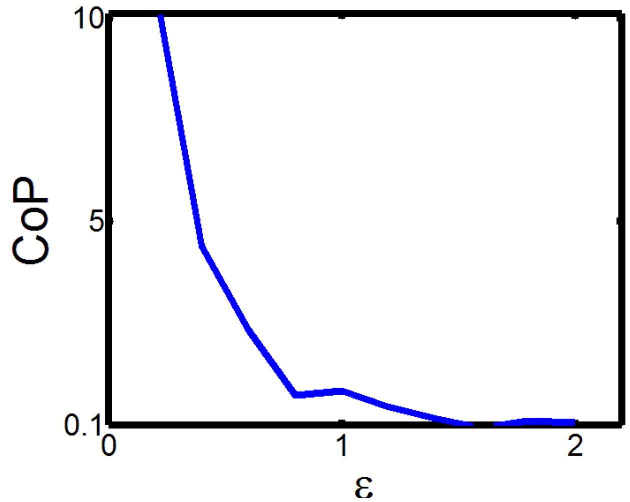
CoP of Linear Distributed Control system

- **Cost of Privacy**: $O(\frac{T^3}{N^2\epsilon^2})$

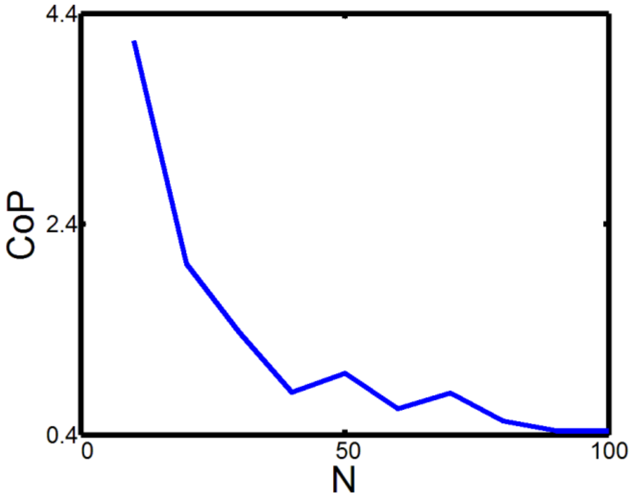
- The strategy works for system with many short-lived agents
- The cost of privacy is low for systems with large number of agents

- Improvement: protecting several waypoint instead of the whole desired trajectory $CoP \sim O(\frac{k^3}{N^2\epsilon^2})$

Cost of Privacy



Cost v.s. (Decreasing) Privacy



Cost v.s. (Increasing) Number of agents

Conclusion

- A framework for studying the cost of differential privacy for distributed control systems.
- A communication strategy to guarantee differential privacy.
- A linear system with quadratic cost is specified
  - Cost of privacy has the order $O(\frac{T^3}{N^2\epsilon^2})$ for stable dynamics, and grows exponentially otherwise.